Colloid Polym Sci 277:563–569 (1999)

ORIGINAL CONTRIBUTION

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Electrostatic interaction between a cylinder and a planar surface

Received: 22 September 1998
Accepted in revised form: 27 January 1999

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Abstract An exact analytical expression for the potential energy of the electrostatic interaction between a plate-like particle 1 and a cylindrical particle 2 of radius a_2 immersed in an electrolyte solution of Debye–Hückel parameter κ is derived on the basis of the linearized Poisson–Boltzmann equation without recourse to Derjaguin's approximation. Both particles may have either constant surface potential or constant surface charge density. In the limit of $\kappa a_2 \to 0$, in particular, the interaction between a plate with zero surface charge density and a cylinder having constant surface charge density becomes identical to the usual image interaction between a line charge (a charged rod of infinitesimal thickness) and an uncharged plate.

Key words Cylinder/surface interaction – Electrostatic interaction – Image interaction

Introduction

In the DLVO theory of colloid stability [1, 2] electrostatic interactions or forces acting between charged colloidal particles play an essential role in determining the behavior of colloidal suspensions. In order to calculate the potential energy of electrostatic interactions, one needs to solve the Poisson–Boltzmann equation for the electric potential distribution in the system of interacting particles. Recently we have shown [3–11] that the linearized Poisson–Boltzmann equation (the Debye–Hückel equation) can be solved exactly for two interacting charged particles to derive explicit exact analytical expressions for the interaction energy for these systems. The energy expressions obtained [3–11] do not use Derjaguin's approximation method [12] and are thus applicable to all values of the particle size and the interparticle separations provided that potentials are low.

The purpose of the present paper is to derive an expression for the potential energy of the electrostatic interaction between a cylindrical particle and a planar surface. Although we have already derived an expression for the interaction energy between two parallel cylinders in a previous paper [9], mathematically it is difficult to take the limit of infinitely large radius of one of the cylinders in this energy expression (in this limit this cylinder becomes a plate), since the cylindrical coordinate is used for both cylinders. In the present paper, the cylindrical coordinate and the Cartesian coordinate are both used.

Interaction between a plate and a cylinder, both at constant surface potential

Linearized Poisson–Boltzmann equation

Consider a planar plate (of semi-infinite thickness) carrying a constant surface potential ψ_{o1} (plate 1) and a charged cylinder of radius a_2 carrying a constant surface potential ψ_{o2} (cylinder 2), separated by a distance H between their surfaces, immersed in an electrolyte solution. The axis of the cylinder is parallel to the plate surface. We employ both a Cartesian coordinate system and a cylindrical coordinate system, as in Fig. 1, in which the origin of the Cartesian coordinate system, O_1, is located at the surface of plate 1 and the origin of the cylindrical coordinate system is located at the center O_2 of cylinder 2. Here z and r, respectively,

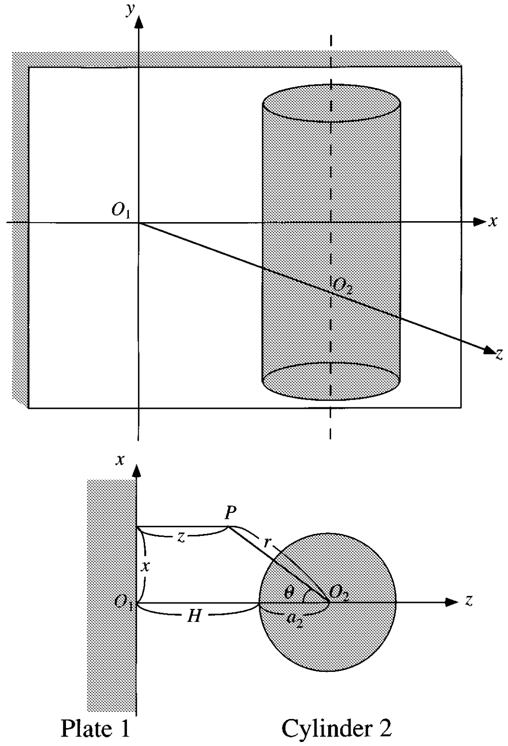

Fig. 1 Interaction between a charged plate 1 and a cylinder 2 of radius a_2 at a separation H between their surfaces. The cylinder axis is parallel to the plate surface

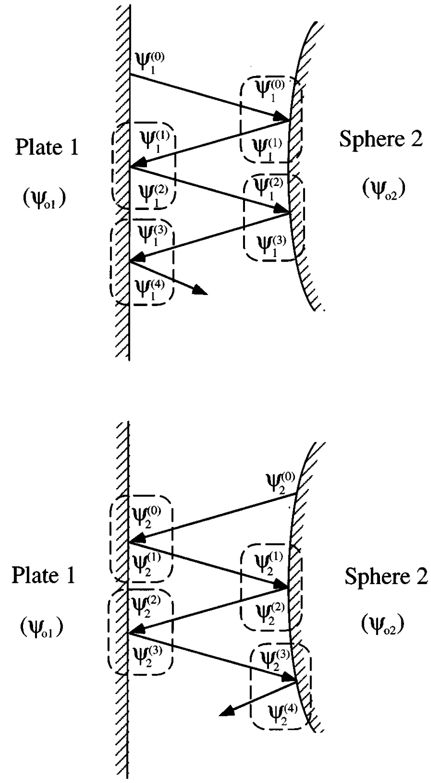

Fig. 2 The unperturbed potentials $\psi_1^{(0)}$, $\psi_2^{(0)}$ and the correction terms $\psi_1^{(k)}$ and $\psi_2^{(k)}$ $(k = 1, 2, \ldots)$ for the constant surface potential case

represent the distance measured from any point P to the surface of plate 1 and to the center O_2 of cylinder 2, x is the distance between the point P and the z-axis (the line O_1O_2), which is perpendicular to the surface of plate 1, and θ is the angle between PO_2 and the z-axis. We first treat the case where the surface potentials of plate 1 and cylinder 2 both remain constant during interaction independent of H.

We assume that the electric potential ψ at any point in the solution phase, measured relative to the bulk solution phase (where ψ is set equal to zero), is low enough to obey the following linearized Poisson–Boltzmann equation

$$\Delta\psi = \kappa^2\psi \tag{1}$$

where κ is the Debye–Hückel parameter of the electrolyte solution. As a result of the symmetry of the system, the potential ψ is a function of x and z: $\psi = \psi(x, z)$ or a function of r and θ: $\psi = \psi(r, \theta)$. The boundary conditions for ψ are

$$\psi(x, 0) = \psi_{o1} = \text{constant (on plate1)} , \tag{2}$$

$$\psi(a_2, \theta) = \psi_{o2} = \text{constant (oncylinder 2)} . \tag{3}$$

We can write the solution to Eq. (1) subject to the boundary conditions (2) and (3) in the following form (Fig. 2)

$$\begin{aligned}\psi &= \psi_1^{(0)} + \psi_2^{(0)} \\ &\quad + \left[\psi_1^{(1)} + \psi_1^{(2)} + \psi_1^{(3)} + \cdots + \psi_1^{(2\nu)} + \psi_1^{(2\nu+1)} + \cdots\right] \\ &\quad + \left[\psi_2^{(1)} + \psi_2^{(2)} + \psi_2^{(3)} + \cdots + \psi_2^{(2\nu)} + \psi_2^{(2\nu+1)} + \cdots\right] .\end{aligned} \tag{4}$$

Zeroth-order terms

As the zeroth-order terms $\psi_1^{(0)}(z)$ and $\psi_2^{(0)}(r)$, we choose the unperturbed potentials produced by plate 1 and cylinder 2 in the absence of interaction (that is, when they are isolated at infinite H), which are functions of only z and r, respectively, viz.,

$$\psi_1^{(0)}(z) = \psi_{o1} \exp(-\kappa z), \quad z \geq 0 \tag{5}$$

$$\psi_2^{(0)}(r) = \psi_{o2} \frac{K_0(\kappa r)}{K_0(\kappa a_2)}, \quad r \geq a_2 \tag{6}$$

where $K_n(x)$ is the modified Bessel function of the second kind. Note that $\psi_1^{(0)}(z)$ and $\psi_2^{(0)}(r)$, respectively, satisfy the boundary conditions (2) and (3).

First-order terms

We construct the first-order terms $\psi_1^{(1)}$ and $\psi_2^{(1)}$ as follows. First we start with the unperturbed potential $\psi_1^{(0)}(z)$, which satisfies the boundary condition (2) on plate 1. The boundary condition (3) on cylinder 2, on the other hand, which has already been satisfied by $\psi_2^{(0)}(r)$, is now violated, since $\psi_1^{(0)}(z)$ gives rise to a nonzero value $\psi_1^{(0)}(a_2,\theta)$ on cylinder 2. We thus construct the first-order term $\psi_1^{(1)}$ so as to cancel $\psi_1^{(0)}(a_2,\theta)$ on cylinder 2, viz.,

$$\psi_1^{(0)}(a_2,\theta)+\psi_1^{(1)}(a_2,\theta)=0 \quad \text{(on cylinder 2)} \ . \tag{7}$$

In order to obtain $\psi_1^{(1)}$, it is convenient to derive an alternative expression for $\psi_1^{(0)}(z)$ on the basis of the (r,θ) coordinate system, viz.,

$$\psi_1^{(0)}(r,\theta)=\psi_{o1}\exp[-\kappa(a_2+H)] \times \sum_{n=-\infty}^{\infty} I_n(\kappa r)\cos(n\theta) \ , \tag{8}$$

where $I_n(x)$ is the modified Bessel function of the first kind and we have used the following formula [13]

$$\exp(\kappa r\cos\theta)\sum_{n=-\infty}^{\infty} I_n(kr)\cos(n\theta) \ , \tag{9}$$

which leads to Eq. (8), if one puts $r\cos\theta = H+a_2-z$. Therefore, we can find that $\psi_1^{(1)}$ must take the form

$$\psi_1^{(1)}(r,\theta)=\psi_{o1}\exp[-\kappa(a_2+H)] \times \sum_{n=-\infty}^{\infty} G_n(2)K_n(\kappa r)\cos(n\theta) \ , \tag{10}$$

with

$$G_n(2)=-\frac{I_n(\kappa a_2)}{K_n(\kappa a_2)} \ . \tag{11}$$

Next we start with the unperturbed potential $\psi_2^{(0)}(r)$, which satisfies the boundary condition (3) on cylinder 2. The boundary condition (2) on plate 1, on the other hand, which has already been satisfied $\psi_1^{(0)}(z)$, is now violated, since $\psi_2^{(0)}(r)$ gives rise to a nonzero value $\psi_2^{(0)}(x,0)$ on plate 1. We thus construct the first-order term $\psi_2^{(1)}$ so as to cancel $\psi_2^{(0)}(x,0)$ on plate 1, viz.,

$$\psi_2^{(0)}(x,0)+\psi_2^{(1)}(x,0)=0 \quad \text{(on plate 1)} \ . \tag{12}$$

In order to obtain $\psi_2^{(1)}$, it is convenient to derive an alternative expression for $\psi_2^{(0)}(r)$ on the basis of the (x,z) coordinate system, viz.,

$$\psi_2^{(0)}(x,z)=\frac{\psi_{o2}}{K_0(\kappa a_2)}\int_0^\infty \frac{1}{\sqrt{k^2+\kappa^2}}\cos(kx) \times \exp\left[-\sqrt{k^2+\kappa^2}\,(H+a_2-z)\right]\,\mathrm{d}k \ , \tag{13}$$

where we have used the following formula [13]

$$K_0(\kappa r)=\int_0^\infty \frac{1}{\sqrt{k^2+\kappa^2}}\cos(kx) \times \exp\left[-\sqrt{k^2+\kappa^2}\,(H+a_2-z)\right]\,\mathrm{d}k \ . \tag{14}$$

Therefore, $\psi_2^{(1)}$ must take the form

$$\psi_2^{(1)}(x,z)=-\frac{\psi_{o2}}{K_0(\kappa a_2)}\int_0^\infty \frac{1}{\sqrt{k^2+\kappa^2}}\cos(kx) \times \exp\left[-\sqrt{k^2+\kappa^2}\,(H+a_2+z)\right]\,\mathrm{d}k \ . \tag{15}$$

Second- and Higher-order terms

In a similar way, we can construct the second-order terms $\psi_1^{(2)}$ and $\psi_2^{(2)}$ so as to cancel $\psi_1^{(1)}$ on plate 1 and $\psi_2^{(1)}$ on cylinder 2, respectively, viz.,

$$\psi_1^{(1)}(x,0)+\psi_1^{(2)}(x,0)=0, \quad \text{(on plate 1)} \ , \tag{16}$$

$$\psi_2^{(1)}(a_2,\theta)+\psi_2^{(2)}(a_2,\theta)=0, \quad \text{(on cylinder 2)} \ . \tag{17}$$

In order to find $\psi_1^{(2)}$, we rewrite $\psi_1^{(1)}$ (Eq. 10) on the basis of the $(x,\ z)$ coordinate system, viz.,

$$\psi_1^{(1)}(x,z)=\psi_{o1}\exp[-\kappa(H+a_2)]\sum_{n=0}^{\infty} G_n(2) \times \int_0^\infty \frac{1}{\sqrt{k^2+\kappa^2}} T_n\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right)\cos(kx) \times \exp\left[-\sqrt{k^2+\kappa^2}\,(H+a_2-z)\right]\,\mathrm{d}k \ , \tag{18}$$

where $T_n(x)$ is the nth order Tchebycheff's polynomial. Here we have used the following formula

$$K_n(\kappa r)\cos(n\theta)=\int_0^\infty \frac{1}{\sqrt{k^2+\kappa^2}} T_n\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right) \times \cos(kr\sin\theta)\exp\left[-\sqrt{k^2+\kappa^2}\,r\cos\theta\right]\,\mathrm{d}k \tag{19}$$

which leads to Eq. (18), if one puts $r\sin\theta = x$ and $r\cos\theta = H + a_2 - z$. Thus we find that $\psi_1^{(2)}$ is given by

$$\psi_1^{(2)}(x,z) = -\psi_{o1}\exp[-\kappa(H+a_2)]\sum_{n=-\infty}^{\infty} G_n(2) \times \int_0^{\infty} \frac{1}{\sqrt{k^2+\kappa^2}} T_n\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right)\cos(kx) \times \exp\left[-\sqrt{k^2+\kappa^2}\,(H+a_2-z)\right] \mathrm{d}k \ . \quad (20)$$

Similarly, in order to find $\psi_2^{(2)}$, we rewrite $\psi_2^{(1)}$ (Eq. 15) in the (r,θ) coordinate system. We use the following formula

$$\exp\left(\sqrt{k^2+\kappa^2}\,r\cos\theta\right)\cos(kr\sin\theta) = \sum_{n=-\infty}^{\infty} T_n\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right) I_n(\kappa r)\cos(n\theta) \ . \quad (21)$$

By noting that $r\cos\theta = H + a_2 + z$ and $r\sin\theta = x$, Eq. (15) can be rewritten as

$$\psi_2^{(1)}(r,\theta) = -\frac{\psi_{o2}}{K_0(\kappa a_2)}\sum_{n=-\infty}^{\infty}\beta_{n0} I_n(\kappa r)\cos(n\theta) \ , \quad (22)$$

with

$$\beta_{n0} = \int_0^{\infty} \frac{\exp\left[-2\sqrt{k^2+\kappa^2}(H+a_2)\right]}{\sqrt{k^2+\kappa^2}} T_n\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right) \mathrm{d}k = \int_1^{\infty} \frac{\exp[-2\kappa t(H+a_2)]}{\sqrt{t^2-1}} T_n(t)\ \mathrm{d}t = K_n[2\kappa(H+a_2)] \ . \quad (23)$$

We thus obtain for $\psi_2^{(2)}$

$$\psi_2^{(2)}(r,\theta) = -\frac{\psi_{o2}}{K_0(\kappa a_2)}\sum_{n=-\infty}^{\infty}\beta_{n0} G_n(2) K_n(\kappa r)\cos(n\theta) \ . \quad (24)$$

By repeating the above procedure one can construct $\psi_1^{(2v-2)}$, $\psi_1^{(2v-1)}$, $\psi_2^{(2v-2)}$, and $\psi_2^{(2v-1)}$ $(v = 1, 2, \ldots)$ that satisfy

$$\psi_1^{(2v-1)}(x,0) + \psi_1^{(2v)}(x,0) = 0, \quad \text{(on plate 1)} \ , \quad (25)$$

$$\psi_1^{(2v-2)}(a_2,\theta) + \psi_1^{(2v-1)}(a_2,\theta) = 0, \quad \text{(on cylinder 2)} \ , \quad (26)$$

$$\psi_2^{(2v-2)}(x,0) + \psi_2^{(2v-1)}(x,0) = 0, \quad \text{(on plate 1)} \ , \quad (27)$$

$$\psi_2^{(2v-1)}(a_2,\theta) + \psi_2^{(2v)}(a_2,\theta) = 0, \quad \text{(on cylinder 2)} \ , \quad (28)$$

so that the boundary conditions (2) and (3) are satisfied, viz.,

$$\psi(x,0) = \psi_1^{(0)}(0) + \sum_{v=1}^{\infty}\left[\psi_1^{(2v-1)}(x,0) + \psi_1^{(2v)}(x,0)\right] + \sum_{v=1}^{\infty}\left[\psi_2^{(2v-2)}(x,0) + \psi_2^{(2v-1)}(x,0)\right] = \psi_{o1} \quad \text{(on plate 1)} \ , \quad (29)$$

$$\psi(a_2,\theta) = \psi_2^{(0)}(a_2) + \sum_{v=1}^{\infty}\left[\psi_1^{(2v-2)}(a_2,\theta) + \psi_1^{(2v-1)}(a_2,\theta)\right] + \sum_{v=1}^{\infty}\left[\psi_2^{(2v-1)}(a_2,\theta) + \psi_2^{(2v)}(a_2,\theta)\right] = \psi_{o2} \quad \text{(on cylinder 2)} \ . \quad (30)$$

Potential energy of electrostatic interaction

The free energy $F(H)$ per unit length of the present system at low potentials can be expressed as [2]

$$F(H) = -\frac{1}{2}\int_{S_1}\sigma_1\psi_{o1}\,\mathrm{d}S_1 - \frac{1}{2}\int_{S_2}\sigma_2\psi_{o2}\,\mathrm{d}S_2 = -\frac{1}{2}\psi_{o1}\int_0^{\infty}\sigma_1(x)\,\mathrm{d}x - \frac{1}{2}\psi_{o2}\int_0^{2\pi}\sigma_2(\theta)a\ \mathrm{d}\theta \ , \quad (31)$$

where the integral is carried out over the surface S_1 of plate 1 and the surface S_2 of cylinder 2. The surface charge densities σ_1 on plate 1 and σ_2 on cylinder 2 are related to the potential derivative at their surfaces by

$$\sigma_1(x) = -\varepsilon\varepsilon_o \left.\frac{\partial\psi}{\partial z}\right|_{z=0^+} \ , \quad (32)$$

$$\sigma_2(\theta) = -\varepsilon\varepsilon_o \left.\frac{\partial\psi}{\partial r}\right|_{r=a_2^+} \ , \quad (33)$$

where ε is the relative permittivity of the electrolyte solution and ε_o is the permittivity of a vacuum. Substituting the expressions obtained for the potential distribution into Eq. (31), we obtain the required result for the interaction energy per unit length between plate 1 and cylinder 2 at constant surface potential, viz.,

$$V(H) = 2\pi\varepsilon\varepsilon_o\psi_{o1}\psi_{o2}\frac{\exp[-\kappa(H+a_2)]}{K_0(\kappa a_2)} + \pi\varepsilon\varepsilon_o\psi_{o1}^2\exp[-2\kappa(H+a_2)]\sum_{n=-\infty}^{\infty}G_n(2)$$

$$
\begin{aligned}
&- \pi\varepsilon\varepsilon_o \psi_{o2}^2 \frac{\beta_{00}}{[K_0(\kappa a_2)]^2} \\
&- \pi\varepsilon\varepsilon_o \psi_{o1}\psi_{o2} \frac{\exp[-\kappa(H+a_2)]}{K_0(\kappa a_2)} \\
&\times \sum_{n=-\infty}^{\infty} (\beta_{0n} + \beta_{n0}) G_n(2) \\
&- \pi\varepsilon\varepsilon_o \psi_{o1}^2 \exp[-2\kappa(H+a_2)] \\
&\times \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \beta_{nm} G_n(2) G_m(2) \\
&+ \pi\varepsilon\varepsilon_o \psi_{o2}^2 \frac{1}{[K_0(\kappa a_2)]^2} \sum_{n=-\infty}^{\infty} \beta_{0n}\beta_{n0} G_n(2) \\
&+ \pi\varepsilon\varepsilon_o \psi_{o1}\psi_{o2} \frac{\exp[-\kappa(H+a_2)]}{K_0(\kappa a_2)} \\
&\times \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \beta_{nm}(\beta_{0n} + \beta_{n0}) G_n(2) G_m(2) + \cdots \\
&- \pi\varepsilon\varepsilon_o \psi_{o1}\psi_{o2} \frac{\exp[-\kappa(H+a_2)]}{K_0(\kappa a_2)} \\
&\times \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_\nu=-\infty}^{\infty} (-1)^{\nu-1} \\
&\times \beta_{n_1 n_2}\beta_{n_2 n_3} \cdots \beta_{n_{\nu-1} n_\nu} (\beta_{0 n_1} + \beta_{n_\nu 0}) \\
&\times G_{n_1}(2) G_{n_2}(2) \cdots G_{n_\nu}(2) \\
&+ \pi\varepsilon\varepsilon_o \psi_{o1}^2 \exp[-2\kappa(H+a_2)] \\
&\times \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_\nu=-\infty}^{\infty} (-1)^{\nu-1} \\
&\times \beta_{n_1 n_2}\beta_{n_2 n_3} \cdots \beta_{n_{\nu-1} n_\nu} \\
&\times G_{n_1}(2) G_{n_2}(2) \cdots G_{n_\nu}(2) \\
&+ \pi\varepsilon\varepsilon_o \psi_{o2}^2 \frac{1}{[K_0(\kappa a_2)]^2} \\
&\times \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_\nu=-\infty}^{\infty} (-1)^{\nu-1} \\
&\times \beta_{n_1 n_2}\beta_{n_2 n_3} \cdots \beta_{n_{\nu-1} n_\nu} \beta_{0 n_1} \beta_{n_\nu 0} \\
&\times G_{n_1}(2) G_{n_2}(2) \cdots G_{n_\nu}(2) + \cdots \quad , \qquad (34)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{nm} &= \int_0^\infty \frac{\exp\left[-2\sqrt{k^2+\kappa^2}(H+a_2)\right]}{\sqrt{k^2+\kappa^2}} \\
&\quad \times T_n\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right) T_m\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right) \mathrm{d}k \\
&= \int_1^\infty \frac{\exp[-2\kappa t(H+a_2)]}{\sqrt{t^2-1}} T_n(t) T_m(t)\, \mathrm{d}t \quad . \qquad (35)
\end{aligned}
$$

For the special case where $n = 0$ or $m = 0$, Eq. (35) reduces to Eq. (23).

Interaction between a plate and a cylinder, both at constant surface charge density

We next consider the case where the surface charge densities of plate 1 and cylinder 2 remain constant (Fig. 3). In this case one must take into account the electric fields induced within the interacting particles. The Laplace equations for the internal regions of plate 1 and cylinder 2 as well as the linearized Poisson–Boltzmann equations for the outside of plate 1 and cylinder 2 must be considered:

$$\Delta\psi = \kappa^2\psi, \quad \text{outsidte plate 1 and cylinder 2} \ , \qquad (36)$$

$$\Delta\psi_{in1} = 0, \quad \text{inside plate 1} \ , \qquad (37)$$

$$\Delta\psi_{in2} = 0, \quad \text{inside cylinder 2} \ , \qquad (38)$$

The boundary conditions for ψ, ψ_{in1} and ψ_{in2} are thus

$$\psi_{in1}(x, 0^-) = \psi(x, 0^+) \ , \qquad (39)$$

$$\varepsilon_1 \frac{\partial \psi_{in1}}{\partial z}\bigg|_{z=0^-} - \varepsilon \frac{\partial \psi}{\partial z}\bigg|_{z=0^+} = \frac{\sigma_1}{\varepsilon_o} \qquad (40)$$

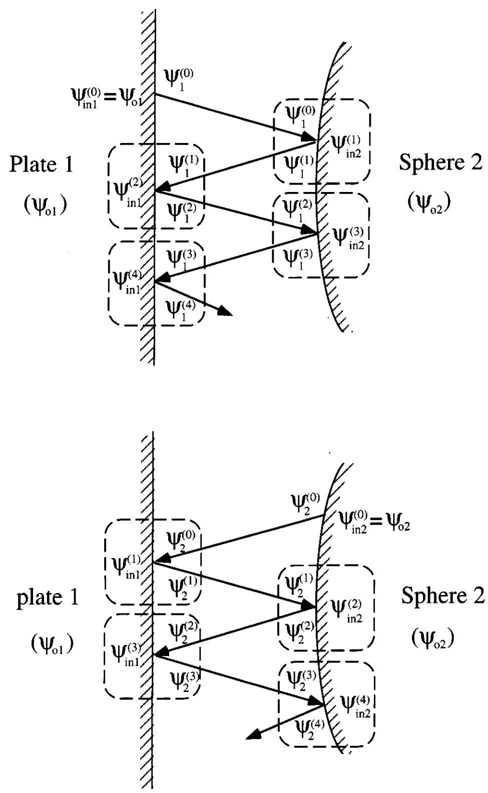

Fig. 3 The unperturbed potentials $\psi_1^{(0)}$, $\psi_2^{(0)}$, $\psi_{in1}^{(0)}$, and $\psi_{in2}^{(0)}$ and the correction terms $\psi_1^{(k)}$, $\psi_2^{(k)}$, $\psi_{in1}^{(k)}$, and $\psi_{in2}^{(k)}$ $(k = 1, 2, \ldots)$ for the constant surface charge density case

at the surface of plate 1, and

$$\psi_{in2}(a_2^-,\theta) = \psi(a_2^+,\theta) \ , \tag{41}$$

$$\varepsilon_2 \frac{\partial \psi_{in2}}{\partial r}\bigg|_{r=a_2^-} - \varepsilon \frac{\partial \psi}{\partial r}\bigg|_{r=a_2^+} = \frac{\sigma_2}{\varepsilon_o} \tag{42}$$

at the surface of cylinder 2. Here ε_1 and ε_2 are, respectively, the relative permittivities of plate 1 and cylinder 2, and the unperturbed surface potentials ψ_{o1} and ψ_{o2} are related to the surface charge densities σ_1 of plate 1 and σ_2 of cylinder 2 by

$$\sigma_1 = \varepsilon\varepsilon_0 \kappa \psi_{o1} \ , \tag{43}$$

$$\sigma_2 = \varepsilon\varepsilon_0 \kappa \psi_{o2} \frac{K_1(\kappa a_2)}{K_0(\kappa a_2)} \ . \tag{44}$$

The free energy $F(H)$ of the present system at low potentials is given by [2]

$$\begin{aligned} F(H) &= +\frac{1}{2}\int_{S_1} \sigma_1 \psi_{o1}\, dS_1 + \frac{1}{2}\int_{S_2} \sigma_2 \psi_{o2}\, dS_2 \\ &= +\frac{1}{2}\sigma_1 \int_0^\infty \psi_1(x,0)\, dx \\ &\quad + \frac{1}{2}\sigma_2 \int_0^{2\pi} \psi_2(a_2,\theta) a_2 \, d\theta \ . \end{aligned} \tag{45}$$

It can be shown that the required result for the interaction energy per unit length between plate 1 and cylinders 1 and 2 at constant surface charge density is obtained by replacing $G_n(2)$ and β_{nm} in Eq. (36) by $H_n(2)$ and $-\gamma_{nm}$, respectively, which are given by

$$H_n(2) = -\frac{I_n'(\kappa a_2) - (\varepsilon_2|n|/\varepsilon\kappa a_2) I_n(\kappa a_2)}{K_n'(\kappa a_2) - (\varepsilon_2|n|/\varepsilon\kappa a_2) K_n(\kappa a_2)} \tag{46}$$

$$\begin{aligned} \gamma_{nm} &= -\int_0^\infty \frac{\varepsilon_1 k - \varepsilon\sqrt{k^2+\kappa^2}}{\varepsilon_1 k + \varepsilon\sqrt{k^2+\kappa^2}} \frac{\exp[-2\sqrt{k^2+\kappa^2}(H+a_2)]}{\sqrt{k^2+\kappa^2}} \\ &\quad \times T_n\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right) T_m\left(\frac{\sqrt{k^2+\kappa^2}}{\kappa}\right) dk \\ &= \int_1^\infty \frac{\varepsilon t - \varepsilon_1\sqrt{t^2-1}}{\varepsilon t + \varepsilon_1\sqrt{t^2-1}} \frac{\exp[-2\kappa t(H+a_2)]}{\sqrt{t^2-1}} \\ &\quad \times T_n(t) T_m(t)\, dt \ . \end{aligned} \tag{47}$$

Results and discussion

We have derived explicit exact analytical expressions for the interaction energy per unit length between a plate-like particle and a cylindrical particle on the basis of the linearized Poisson–Boltzmann equation. When the surface potentials of the interacting particles remain constant, the interaction energy is given by Eq. (34). If, on the other hand, the surface charge densities of the interacting particles remain constant, then the interaction energy is given by replacing $G_n(2)$ and β_{nm} in Eq. (34) by $H_n(2)$ and $-\gamma_{nm}$, respectively. An expression for the interaction energy for the mixed cases where either plate 1 or cylinder 2 has a constant surface potential and the other has a constant surface charge density can also be obtained. It can easily be shown that when plate 1 has a constant surface potential and cylinder 2 has a constant surface charge density, the interaction energy is given by Eq. (34) with $G_n(2)$ replaced by $H_n(2)$. When plate 1 has a constant surface charge density and cylinder 2 has a constant surface potential, the interaction energy is given by Eq. (34) with β_{nm} replaced by $-\gamma_{nm}$. Also we have confirmed numerically that the results obtained in the present paper agree with the previous results when the radius of either cylinder tends to infinity [9].

The first term in Eq. (34) corresponds to the interaction energy obtained by the linear superposition approximation. The second term in Eq. (34), which depends only on the unperturbed surface potential ψ_{o1} of plate 1 and does not depend on the unperturbed surface potential ψ_{o2} of cylinder 2, corresponds to the image interaction between plate 1 and its image with respect to cylinder 2. The third term in Eq. (34) corresponds to the image interaction between cylinder 2 and its image with respect to plate 1.

Consider the case where $\kappa a_2 \gg 1$. For the constant surface potential case, Eq. (34) becomes

$$\begin{aligned} V(H) &= 2\sqrt{2}\pi\varepsilon\varepsilon_o\sqrt{\kappa a_2}[\psi_{o1}\psi_{o2}\exp(-\kappa H) \\ &\quad - \frac{1}{2\sqrt{2}}\left(\psi_{o1}^2 + \psi_{o2}^2\sqrt{\frac{a_2}{a_2+H}}\right)\exp(-2\kappa H)] \\ &\quad + O[\exp(-3\kappa H)] \ . \end{aligned} \tag{48}$$

For the constant surface charge density case, if ε_1 and ε_2 are finite, we obtain

$$\begin{aligned} V(H) &= 2\sqrt{2}\pi\varepsilon\varepsilon_o\sqrt{\kappa a_2}[\psi_{o1}\psi_{o2}\exp(-\kappa H) \\ &\quad + \frac{1}{2\sqrt{2}}\psi_{o1}^2\left(1 - \frac{2\varepsilon_2}{\varepsilon}\frac{1}{\sqrt{\pi\kappa a_2}}\right)\exp(-2\kappa H) \\ &\quad + \frac{1}{2\sqrt{2}}\psi_{o2}^2\sqrt{\frac{a_2}{a_2+H}}\left(1 - \frac{2\varepsilon_1}{\varepsilon}\frac{1}{\sqrt{\pi\kappa(a_2+H)}}\right) \\ &\quad \times \exp(-2\kappa H)] + O[\exp(-3\kappa H)], \\ &\quad \left(\frac{\varepsilon_1}{\varepsilon}\frac{1}{\sqrt{\pi\kappa(a_2+H)}} \ll 1, \frac{\varepsilon_2}{\varepsilon}\frac{1}{\sqrt{\pi\kappa a_2}} \ll 1\right) \end{aligned} \tag{49}$$

and if ε_1 and ε_2 are infinity (plate 1 and cylinder 2 are both metallic),

$$V(H) = 2\sqrt{2\pi}\varepsilon\varepsilon_o\sqrt{\kappa a_2}\Big[\psi_{o1}\psi_{o2}\exp(-\kappa H) - \frac{1}{2\sqrt{2}}\left(\psi_{o1}^2 + \psi_{o2}^2\sqrt{\frac{a_2}{a_2+H}}\right)\exp(-2\kappa H)] + O[\exp(-3\kappa H)\Big], \quad (\varepsilon_1 = \infty, \varepsilon_2 = \infty) \ . \quad (50)$$

Similarly we can obtain the corresponding energy expressions for the mixed cases.

Finally we compare the image interactions appearing in the present theory with the usual image interaction between a line charge and a plate by taking the limit of $\kappa a_2 \to 0$. This is possible because the energy expressions obtained are valid for all values of the cylinder radius. For this purpose we consider the case where the surface charge density of plate 1 is always zero ($\psi_{o1} = 0$) and $\kappa a_2 \to 0$, since the usual image interaction refers to a point charge interacting with an uncharged plate. In the case of $\psi_{o1} = 0$, the interaction energy $V(H)$ becomes

$$V(H) = \pi\varepsilon\varepsilon_o\psi_{o2}^2\frac{1}{[K_0(\kappa a_2)]^2}\Big[\gamma_{00} + \sum_{n=-\infty}^{\infty}\gamma_{0n}\gamma_{n0}H_n(2) + \cdots + \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_v=-\infty}^{\infty}\gamma_{n_1n_2}\gamma_{n_2n_3}\cdots\gamma_{n_{v-1}n_v} \times \gamma_{0n_1}\gamma_{n_v0}H_{n_1}(2)H_{n_2}(2)\cdots H_{n_v}(2) + \cdots\Big] \ . \quad (51)$$

In the limit $\kappa a_2 \to 0$, all the terms on the right-hand side except the first term vanish. We introduce the total charge $Q \equiv 2\pi a\sigma_2$ on cylinder 2 per unit length (i.e., the line charge density of cylinder 2), which is related to the unperturbed surface potential ψ_{o2} of cylinder 2 by

$$\psi_{o2} = \frac{Q}{2\pi\varepsilon\varepsilon_0\kappa a_2}\frac{K_0(\kappa a_2)}{K_1(\kappa a_2)} \ . \quad (52)$$

For the special cases where $\varepsilon_1 = 0$ or $\varepsilon_1 = \infty$, γ_{00} tends to

$$\gamma_{00} = \begin{cases} K_0(2\kappa H), & (\varepsilon_1 = 0) \\ -K_0(2\kappa H), & (\varepsilon_1 = \infty) \ . \end{cases} \quad (53)$$

Thus we have

$$V(H) = \begin{cases} \frac{Q^2}{4\pi\varepsilon\varepsilon_0}K_0(2\kappa H), & (\varepsilon_1 = 0) \\ -\frac{Q^2}{4\pi\varepsilon\varepsilon_0}K_0(2\kappa H), & (\varepsilon_1 = \infty) \ . \end{cases} \quad (54)$$

This is the screened image interaction between a line charge and an uncharged plate, both immersed in an electrolyte solution of Debye–Hückel parameter κ. We see that in the former case ($\varepsilon_1 = 0$) the interaction force is repulsion and in the latter case ($\varepsilon_1 = \infty$) attraction. Further, in the absence of electrolytes ($\kappa \to 0$), we can show from Eq. (51) that the interaction force $-\partial V/\partial H$ per unit length between plate 1 and cylinder 2 with $a_2 \to 0$ is given by

$$-\frac{\partial V}{\partial H} = \frac{Q^2}{4\pi\varepsilon\varepsilon_0}\left(\frac{\varepsilon - \varepsilon_1}{\varepsilon + \varepsilon_1}\right)\frac{1}{H} \quad (55)$$

which agrees exactly with the usual image force per unit length between a line charge and an uncharged plate [14]. Figure 4 illustrates how the image interaction between a cylindrical particle of finite size and a plate calculated from Eq. (51) approaches the usual image interaction between a line charge and a plate for the two cases $\varepsilon_1 = 0$ and $\varepsilon_1 = \infty$ (Eq. 54).

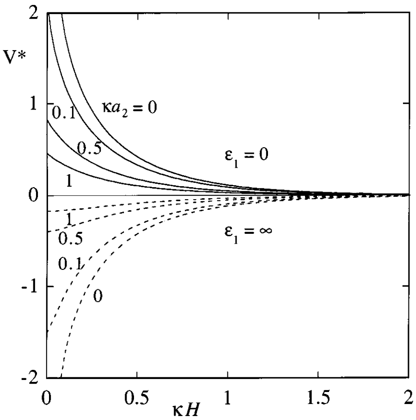

Fig. 4 Reduced potential energy $V^* \equiv 4\pi\varepsilon\varepsilon_o V/Q^2$ of the image interaction per unit length between a cylinder (cylinder 2) of radius a_2 with $\varepsilon_2 = 0$ and a plate (plate 1) as a function of κH for several values of the reduced radius κa_2 of cylinder 2. Calculated with Eq. (51). *Solid lines*: $\varepsilon_1 = 0$; *dashed lines*: $\varepsilon_1 = \infty$ (plate 1 is a metal)

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